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Prime Ideals in Free ℓ-Groups and Free Vector Lattices

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We classify the prime ideals in finitely generated free ℓ-groups and free vector lattices.

1. PRELIMINARIES AND DEFINITIONS

In this paper we give an explicit description of the root system of prime ideals in finitely generated free lattice-ordered abelian groups and free vector lattices. We use the representation of the above structures in terms of piecewise-linear homogeneous functions over real affine space [Bak68, Bey74, Bey77]. Our main technical tool is a derivation operator that allows us to localize the behavior of piecewise-linear functions at given points and along given directions. The resulting description is geometric, in the sense that a prime ideal is characterized as the set of functions that behave locally in a prescribed way.

We obtain a characterization of the root system of prime ideals of finitely generated free vector lattices in terms of the poset of orthonormal tuples of vectors in $\mathbb{R}^n$, ordered by initial segments. The same description holds for free lattice-ordered abelian groups, provided that we restrict to tuples satisfying an added condition of $\mathbb{Z}$-reduction. We prove that, in both cases, the root system of primes and the poset of their germinal ideals are naturally anti-isomorphic, and we give an easy algorithm for computing the isomorphism types of quotients under prime ideals. Our description extends the results in [Dar86] and [Dar95, Sect. 52], and provides alternative proofs. As an immediate consequence of our main theorem, using the

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categorical equivalence between lattice-ordered abelian groups with strong unit and MV-algebras [Mun86], we obtain a complete description of the prime ideal spaces of the finitely generated free MV-algebras.

Our work is essentially self-contained, assuming only basic familiarity with lattice-ordered abelian groups (ℓ-groups, for short), totally ordered abelian groups (o-groups), and vector lattices; see [Fuc63; Bir67; BKW77; AF88; Dar95]. The few definitions we need from the combinatorial theory of cones in $\mathbb{R}^n$ are collected as follows. A *polyhedral* cone σ is the positive span of a finite set of vectors of $\mathbb{R}^n$, i.e., a set of the form $\sigma = \mathbb{R}^+ u_1 + \cdots + \mathbb{R}^+ u_t$. If the u_i's can be completed to a basis of $\mathbb{R}^n$, then σ is *simplicial*, and if the u_i's are in $\mathbb{Z}^n$ and can be completed to a basis of $\mathbb{Z}^n$, then σ is *unimodular*. A fan Σ is a finite complex of polyhedral cones, i.e., a finite set of cones, any two of them intersecting in a common face. For $0 \le k \le n$, $\Sigma(k)$ is the set of k-dimensional cones in Σ. The fan Σ is *complete* if $\bigcup\{\sigma : \sigma \in \Sigma\} = \mathbb{R}^n$, and is *simplicial* (resp. *unimodular*) if all of its cones are simplicial (resp., unimodular). See [Oda88; Ful93; Ewa96] for cones and fans.

DEFINITION 1.1. A continuous function $f\colon \mathbb{R}^n \to \mathbb{R}$ is *piecewise-linear* if it satisfies one of the following equivalent conditions:

1. there exist $f_1, \dots, f_t \in \mathrm{Hom}(\mathbb{R}^n, \mathbb{R})$ such that, for every $u \in \mathbb{R}^n$, $f(u) = f_i(u)$ for some i;
2. there exist $f_1, \dots, f_t \in \mathrm{Hom}(\mathbb{R}^n, \mathbb{R})$ and a complete fan Σ such that $\Sigma(n) = \{\sigma_1, \dots, \sigma_t\}$ and $f = f_i$ over σ_i;
3. f is obtainable from the projection functions $x_i\colon \mathbb{R}^n \to \mathbb{R}$ via the pointwise operations $+$, $-$, 0, $\vee$, $\wedge$, and product by scalars;
4. $f = \bigvee_{1 \le i \le t} \bigwedge_{1 \le j_i \le q_i} f_{ij_i}$, where $f_{ij_i} \in \mathrm{Hom}(\mathbb{R}^n, \mathbb{R})$ for every i, j_i.

DEFINITION 1.2. $\mathrm{FVL}(n)$ is the real vector lattice of all continuous piecewise-linear functions $\mathbb{R}^n \to \mathbb{R}$, with pointwise operations $+$, $-$, 0, $\vee$, $\wedge$, and product by scalars. $F\ell(n)$ is the ℓ-subgroup of $\mathrm{FVL}(n)$ whose elements are all piecewise-linear functions with integer coefficients (these are the functions resulting from Definition 1.1 by replacing $\mathrm{Hom}(\mathbb{R}^n, \mathbb{R})$ with $\mathrm{Hom}(\mathbb{Z}^n, \mathbb{Z})$ and dropping multiplication by scalars).

$\mathrm{FVL}(n)$ and $F\ell(n)$ are, respectively, the free real vector lattice and the free ℓ-group over n generators. The free generators are the projection functions x_i, and in Definition 1.1(3) f is actually freely obtainable from $x_1, \dots, x_n$.

2. DERIVATION OPERATORS

The Euclidean norm of $u \in \mathbb{R}^n$ is denoted by $\|u\|$. We write S^{n-1} for the unit sphere in $\mathbb{R}^n$. The open cone of center $u \neq 0$ and radius $\varepsilon > 0$ is

$$C(u, \varepsilon) = \{v \in \mathbb{R}^n \setminus \{0\} : \|u/\|u\| - v/\|v\|\| < \varepsilon\}.$$

We denote the set of positive real numbers, 0 included, by $\mathbb{R}^+$. We say that $f: \mathbb{R}^n \to \mathbb{R}$ is *linear* on the polyhedral cone μ if $f(ru + sv) = rf(u) + sf(v)$, for every $r, s \in \mathbb{R}^+$ and $u, v \in \mu$. This is equivalent to saying that there exist $w_1, \dots, w_t$ such that $\mu = \mathbb{R}^+ w_1 + \cdots + \mathbb{R}^+ w_t$ and, for every $r_1, \dots, r_t \in \mathbb{R}^+$, $f(\sum r_i w_i) = \sum r_i f(w_i)$.

LEMMA 2.1. *Let* Σ *be a complete fan*, $0 \neq u \in \mathbb{R}^n$. *Then*:

(i) *there exists* $\varepsilon > 0$ *with*

$$C(u, \varepsilon) \subseteq \bigcup \{\sigma \in \Sigma(n) : u \in \sigma\};$$

(ii) *for every* $v \in \mathbb{R}^n$ *there exists* $\lambda_v \in \mathbb{R}^+$ *and* $\sigma \in \Sigma(n)$ *with* $u, \lambda u + v \in \sigma$, *for every* $\lambda \geq \lambda_v$.

Proof. (i) Let

$$K = S^{n-1} \cap \bigcup \{\sigma \in \Sigma(n) : u \notin \sigma\}.$$

For $k = 1, 2, 3, \dots,$ let D_k be the topological closure of $C(u, 1/k) \cap S^{n-1}$; we have

$$\bigcap_{1 \leq k} (K \cap D_k) = \varnothing.$$

Since K is compact, $K \cap D_m = \varnothing$ for some m; take $\varepsilon = 1/m$.

(ii) Let $C(u, \varepsilon)$ be as in (i). Since $C(u, \varepsilon) \cap S^{n-1}$ is an open neighborhood of $u/\|u\|$ in S^{n-1} and

$$\lim_{h \to 0^+} \frac{u + hv}{\|u + hv\|} = \frac{u}{\|u\|},$$

we can fix $h_0 > 0$ such that $u + h_0 v \in C(u, \varepsilon)$. So $u, u + h_0 v \in \sigma$, for some $\sigma \in \Sigma(n)$. Take $\lambda_v = h_0^{-1}$. Then, for every $\lambda \geq \lambda_v$, we have

$$\lambda u + v = (\lambda - \lambda_v)u + \lambda_v(u + h_0 v) \in \sigma.$$

∎

DEFINITION 2.2. Let $f \in \mathrm{FVL}(n)$, $u \in \mathbb{R}^n$. We let $D_u f$ be the function $\mathbb{R}^n \to \mathbb{R}$ defined by

$$D_u f(v) = \lim_{h \to 0^+} \frac{f(u + hv) - f(u)}{h}.$$

LEMMA 2.3. *Fix* $f \in \mathrm{FVL}(n)$, $u \in \mathbb{R}^n$. *Then*:

(i) *for every* $v \in \mathbb{R}^n$ *there exists* $\lambda_v > 0$ *such that the difference* $f(\lambda u + v) - f(\lambda u)$ *is constant and equals* $D_u f(v)$, *as* λ *varies in the set of real numbers* $\geq \lambda_v$. *Moreover*, $D_u f$ *coincides on* $\mathbb{R}u + \mathbb{R}^+ v$ *with one of the* $f_1, \ldots, f_t$ *given by Definition* 1.1(2);

(ii) $D_u f \in \mathrm{FVL}(n)$ *and if* $f \in \mathrm{F}\ell(n)$, *then* $D_u f \in \mathrm{F}\ell(n)$;

(iii) $D_0 f = f$ *and* $D_{ru} f = D_u f$, *for every real number* $r > 0$.

Assume $u \neq 0$. *Then*:

(iv) *there exists* $\varepsilon > 0$ *such that* $D_u f = f$ *on* $C(u, \varepsilon)$;

(v) $D_u f = D_u g$ *iff* $f = g$ *on some* $C(u, \varepsilon)$; *in particular*, $D_u f = 0$ *iff there exists* $\varepsilon > 0$ *with* $f \restriction C(u, \varepsilon) = 0$.

Proof. If $u = 0$, then (i)–(iii) hold trivially; so we assume $u \neq 0$ throughout. Let Σ be a complete fan for f, as in Definition 1.1(2).

(i) Choose $v \in \mathbb{R}^n$; using Lemma 2.1(ii), we find $\sigma_i \in \Sigma(n)$, $f_i \in \mathrm{Hom}(\mathbb{R}^n, \mathbb{R})$, and $\lambda_v > 0$, such that $f = f_i$ on σ_i and $u, \lambda u + v \in \sigma_i$ for every $\lambda \geq \lambda_v$. Then

$$\begin{aligned}
D_u f(v) &= \lim_{h \to 0^+} \frac{f(u + hv) - f(u)}{h} \\
&= \lim_{h \to 0^+} \bigl(f(h^{-1}u + v) - f(h^{-1}u) \bigr) \\
&= \lim_{\lambda \to +\infty} \bigl(f(\lambda u + v) - f(\lambda u) \bigr) \\
&= \lim_{\lambda_v \leq \lambda \to +\infty} \bigl(f_i(\lambda u + v) - f_i(\lambda u) \bigr) \\
&= f_i(v) \\
&= f_i(\lambda u + v) - f_i(\lambda u) \qquad (\text{for every } \lambda) \\
&= f(\lambda u + v) - f(\lambda u) \qquad (\text{for } \lambda \geq \lambda_v).
\end{aligned}$$

If $w = ru + sv$, with $r, s \in \mathbb{R}$ and $s \geq 0$, then one sees easily that $\lambda u + w \in \sigma_i$, for all sufficiently large λ's. Then $f(w) = f_i(w)$ as well, and $D_u f = f_i$ on $\mathbb{R}u + \mathbb{R}^+ v$.

(ii) It suffices to show that $D_u f$ is continuous, since we can then apply Definition 1.1(1). Let $v, w \in \mathbb{R}^n$ and let $\lambda' > 0$ be such that $D_u f(v) = f(\lambda' u + v) - f(\lambda' u)$ and $D_u f(w) = f(\lambda' u + w) - f(\lambda' u)$. Then, when $\|v - w\|$ tends to 0, so does $\|(\lambda' u + v) - (\lambda' u + w)\|$, and therefore $\|D_u f(v) - D_u f(w)\|$ tends to 0, since f is continuous.

(iii) This is straightforward from the definitions.

(iv) If $C(u, \varepsilon)$ is as in Lemma 2.1(i), and $v \in C(u, \varepsilon)$, then we can take $\lambda = \lambda_v = 0$ in (i).

(v) The left-to-right direction follows from (iv). If $f = g$ on $C(u, \varepsilon)$ then, given $v \in \mathbb{R}^n$, we find $\lambda_v > 0$ with $\lambda_v u + v \in C(u, \varepsilon)$, and apply (i). ∎

Note that, unless $u = 0$, the image of $\mathrm{FVL}(n)$ (resp., $\mathrm{F}\ell(n)$) under D_u is a subvector space of $\mathrm{FVL}(n)$ (resp., a subgroup of $\mathrm{F}\ell(n)$), but not an ℓ-subgroup. This is easily seen by taking $n = 1$ and $0 \neq u \in \mathbb{R}^1$; then $D_u x \vee D_u 0 = x \vee 0$, which is not linear on $\mathbb{R}u$. On the other hand, we will prove that:

- $\{D_u f : f \in \mathrm{FVL}(n) \text{ and } f(u) = 0\}$ is a subvector lattice of $\mathrm{FVL}(n)$ (see Theorem 3.8);
- $\{D_u f : f \in \mathrm{FVL}(n)\}$ is a vector lattice in its own right, namely, the vector lattice of germs of functions at u (see Corollary 3.6).

Analogous statements hold for $\mathrm{F}\ell(n)$.

LEMMA 2.4. *Let $g \in \mathrm{FVL}(n)$ and let U be a subspace of $\mathbb{R}^n$ such that, for every $w \in \mathbb{R}^n$, g is linear on $U + \mathbb{R}^+ w$. Then, for every $u \in U$, we have $D_u g = g$. If g satisfies the stronger condition of being constant on $U + w$, for every $w \in \mathbb{R}^n$, then $D_{u+v} g = D_v g$, for every $u \in U$ and $v \in \mathbb{R}^n$.*

Proof. This is a trivial computation. ∎

LEMMA 2.5. *For every $f \in \mathrm{FVL}(n)$, $u, v \in \mathbb{R}^n$, $r \in \mathbb{R}$, we have $D_v D_u f = D_{v+ru} D_u f$.*

Proof. Let $g = D_u f$; by Lemma 2.3(i), g is linear on $\mathbb{R}u + \mathbb{R}^+ w$, for every $w \in \mathbb{R}^n$. Therefore

$$
\begin{aligned}
D_{v+ru} g(w) &= \lim_{h \to 0^+} \frac{g(v + ru + hw) - g(v + ru)}{h} \\
&= \lim_{h \to 0^+} \frac{g(v + hw) + g(ru) - g(v) - g(ru)}{h} \\
&= D_v g(w).
\end{aligned}
$$

∎

LEMMA 2.6. *Let* $g \in \mathrm{FVL}(n)$ *and let* U *be a subspace of* $\mathbb{R}^n$ *such that, for every* $w \in \mathbb{R}^n$, g *is linear on* $U + \mathbb{R}^+ w$. *Then, for every* $w \in \mathbb{R}^n$, *we have* $g \restriction (U + \mathbb{R}^+ w) = D_w g \restriction (U + \mathbb{R}^+ w)$, *and* $D_w g$ *is linear on every* $U + \mathbb{R}w + \mathbb{R}^+ v$.

Proof. The equality $g \restriction (U + \mathbb{R}^+ w) = D_w g \restriction (U + \mathbb{R}^+ w)$ follows from an easy computation. Let $s, r \in \mathbb{R}$, $r \geq 0$; we have

$$\begin{aligned} D_w g(u + sw + rv) &= \lim_{h \to 0^+} \frac{g(w + hu + hsw + hrv) - g(w)}{h} \\ &= g(u) + \lim_{h \to 0^+} \frac{g(w + hsw + hrv) - g(w)}{h} \\ &= D_w g(u) + D_w g(sw + rv) \\ &= D_w g(u) + s D_w g(w) + r D_w g(v), \end{aligned}$$

by Lemma 2.3(i). ∎

COROLLARY 2.7. *Let* $f \in \mathrm{FVL}(n)$, $u_1, \ldots, u_t \in \mathbb{R}^n$. *Then*:

(i) $f(u_1) = D_{u_1} f(u_1) = D_{u_2} D_{u_1} f(u_1) = \cdots = D_{u_t} \cdots D_{u_1} f(u_1)$;

(ii) *for every* $v \in \mathbb{R}^n$, $D_{u_t} \cdots D_{u_1} f$ *is linear on* $\mathbb{R}u_1 + \cdots + \mathbb{R}u_t + \mathbb{R}^+ v$;

(iii) *for every* $u \in \mathbb{R}u_1 + \cdots + \mathbb{R}u_t$, *we have* $D_u D_{u_t} \cdots D_{u_1} f = D_{u_t} \cdots D_{u_1} f$;

(iv) *there exists* $\varepsilon > 0$ *such that* $D_{u_t} \cdots D_{u_1} f = D_{u_{t-1}} \cdots D_{u_1} f$ *on* $\mathbb{R}u_1 + \cdots + \mathbb{R}u_{t-1} + C(u_t, \varepsilon)$.

Proof. Clearly $f(u_1) = D_{u_1} f(u_1)$. By Lemma 2.3(i) $D_{u_1} f$ is linear on every $\mathbb{R}u_1 + \mathbb{R}^+ v$; Lemma 2.6 and induction on t prove (i) and (ii). We infer (iii) from (ii) and Lemma 2.4. About (iv): let $g = D_{u_{t-1}} \cdots D_{u_1} f$. Using Lemma 2.3(iv), take ε such that $g = D_{u_t} g$ on $C(u_t, \varepsilon)$. By (ii), both g and $D_{u_t} g$ are linear on $\mathbb{R}u_1 + \cdots + \mathbb{R}u_{t-1} + \mathbb{R}^+ v$, for every $v \in \mathbb{R}^n$. Since $g = D_{u_t} g$ both on $\mathbb{R}u_1 + \cdots + \mathbb{R}u_{t-1}$ (by Lemma 2.6) and on $C(u_t, \varepsilon)$ (by our choice of ε), the conclusion follows. ∎

LEMMA 2.8. *Let* $f \in \mathrm{FVL}(n)$. *The following are equivalent*:

1. $f \in \mathrm{Hom}(\mathbb{R}^n, \mathbb{R})$;
2. *for every* $u \in \mathbb{R}^n$, $D_u f = f$;
3. *there exists a basis* $u_1, \ldots, u_n$ *of* $\mathbb{R}^n$ *such that* $D_{u_1} f = \cdots = D_{u_n} f = f$;
4. *there exists a basis* $u_1, \ldots, u_n$ *of* $\mathbb{R}^n$ *and some* $g \in \mathrm{FVL}(n)$ *with* $f = D_{u_n} \cdots D_{u_1} g$.

Proof. Clearly (1) $\Rightarrow$ (2) $\Rightarrow$ (3). We obtain (3) $\Rightarrow$ (4) by taking $g = f$. Finally, (4) $\Rightarrow$ (1) by Corollary 2.7(ii). ∎

Of course, D_u distributes over $+$. The behavior of D_u with respect to the lattice operations is given by the following lemma.

LEMMA 2.9. *Let $f, g \in \mathrm{FVL}(n)$, $u \in \mathbb{R}^n$. Then*:

(i) $D_u(f \vee g) = D_u(D_u f \vee D_u g)$;

(ii) *if $f(u) = g(u)$, then $D_u(f \vee g) = D_u f \vee D_u g$*;

(iii) *if $f(u) < g(u)$, then $D_u(f \vee g) = D_u g$.*

Proof. If $u = 0$, then all our claims are trivial. Assume $u \neq 0$; using Lemma 2.3(iv), find $\varepsilon > 0$ such that $f = D_u f$ and $g = D_u g$ on $C(u, \varepsilon)$. Then $f \vee g = D_u f \vee D_u g$ on $C(u, \varepsilon)$, and (i) follows from Lemma 2.3(v). Suppose $f(u) = g(u)$. By Lemma 2.3(i), $D_u f$ and $D_u g$ coincide on $\mathbb{R}u$ and, moreover, both are linear on $\mathbb{R}u + \mathbb{R}^+ v$, for every $v \in \mathbb{R}^n$. Therefore, $D_u f \vee D_u g$ is linear on $\mathbb{R}u + \mathbb{R}^+ v$, for every $v \in \mathbb{R}^n$. By Lemma 2.4, $D_u(D_u f \vee D_u g) = D_u f \vee D_u g$, and we deduce (ii) from (i). Suppose now $f(u) < g(u)$. Since both f and g are continuous, we may assume $f < g$ on some $C(u, \varepsilon)$. Then $f \vee g = g$ on $C(u, \varepsilon)$, and (iii) follows from Lemma 2.3(v). ∎

3. PRIME IDEALS IN FREE VECTOR LATTICES

We denote by $\mathbf{u}, \mathbf{v}, \dots$ orthonormal tuples $\mathbf{u} = (u_1, \dots, u_t)$ of elements of $\mathbb{R}^n$. An unspecified $\mathbf{u}$ has always length t, with $1 \leq t \leq n$. We denote by $\otimes^t \mathbb{R}$ the direct product of t copies of $\mathbb{R}$, ordered lexicographically from left to right. Define

$$\varphi_{\mathbf{u}} \colon \mathrm{FVL}(n) \to \otimes^t \mathbb{R}$$

by

$$\varphi_{\mathbf{u}}(f) = \bigl(f(u_1), D_{u_1} f(u_2), D_{u_2} D_{u_1} f(u_3), \dots, D_{u_{t-1}} \cdots D_{u_1} f(u_t)\bigr).$$

If $g = D_{u_{t-1}} \cdots D_{u_1} f$ and $h = D_{u_t} g$, then by Corollary 2.7(i) we have

$$\varphi_{\mathbf{u}}(f) = (g(u_1), \dots, g(u_t)) = (h(u_1), \dots, h(u_t)).$$

For every tuple $\boldsymbol{\varepsilon} = (\varepsilon_1, \ldots, \varepsilon_t)$ of strictly positive reals, let $S(\mathbf{u}, \boldsymbol{\varepsilon})$ be the cone generated by

$$\left\{ \sum_{j=1}^{i} \varepsilon_j u_j : 1 \le i \le t \right\}.$$

Finally, let

$$\mathfrak{p}_{\mathbf{u}} = \{ f \in \mathrm{FVL}(n) : \text{there exists } \boldsymbol{\varepsilon} \text{ with } f = 0 \text{ on } S(\mathbf{u}, \boldsymbol{\varepsilon}) \}.$$

THEOREM 3.1. *Every $\varphi_{\mathbf{u}}$ is an epimorphism with kernel $\mathfrak{p}_{\mathbf{u}}$.*

Proof. Since every D_u distributes over $+$, it follows that $\varphi_{\mathbf{u}}$ is a homomorphism with respect to the group structure. Let $\varphi_{\mathbf{u}}(f) = (\alpha_1, \ldots, \alpha_t)$, $\varphi_{\mathbf{u}}(g) = (\beta_1, \ldots, \beta_t)$. Without loss of generality, $\varphi_{\mathbf{u}}(f) \vee \varphi_{\mathbf{u}}(g) = \varphi_{\mathbf{u}}(g)$ in $\otimes^t \mathbb{R}$; this means that, for some $0 \le j \le t$, we have $\alpha_{j+1} < \beta_{j+1}$, while $\alpha_i = \beta_i$ for every $i \le j$. Now:

(a) if $j = 0$, then $f(u_1) < g(u_1)$. By Lemma 2.9(iii), $D_{u_1}(f \vee g) = D_{u_1} g$, and so $\varphi_{\mathbf{u}}(f \vee g) = \varphi_{\mathbf{u}}(g)$;

(b) if $1 \le j < t$, then

$$\begin{aligned} f(u_1) &= g(u_1) \\ D_{u_1} f(u_2) &= D_{u_1} g(u_2) \\ \vdots &= \vdots \\ D_{u_{j-1}} \cdots D_{u_1} f(u_j) &= D_{u_{j-1}} \cdots D_{u_1} g(u_j) \\ D_{u_j} \cdots D_{u_1} f(u_{j+1}) &< D_{u_j} \cdots D_{u_1} g(u_{j+1}). \end{aligned} \tag{$*$}$$

From these equalities and Lemma 2.9, we have

$$\begin{aligned} D_{u_1}(f \vee g) &= D_{u_1} f \vee D_{u_1} g \\ D_{u_2} D_{u_1}(f \vee g) &= D_{u_2}\big(D_{u_1} f \vee D_{u_1} g\big) \\ &= D_{u_2} D_{u_1} f \vee D_{u_2} D_{u_1} g \\ \vdots &= \vdots \\ D_{u_j} D_{u_{j-1}} \cdots D_{u_1}(f \vee g) &= D_{u_j}\big(D_{u_{j-1}} \cdots D_{u_1} f \vee D_{u_{j-1}} \cdots D_{u_1} g\big) \\ &= D_{u_j} D_{u_{j-1}} \cdots D_{u_1} f \vee D_{u_j} D_{u_{j-1}} \cdots D_{u_1} g \\ D_{u_{j+1}} D_{u_j} \cdots D_{u_1}(f \vee g) &= D_{u_{j+1}}\big(D_{u_j} \cdots D_{u_1} f \vee D_{u_j} \cdots D_{u_1} g\big) \\ &= D_{u_{j+1}} D_{u_j} \cdots D_{u_1} g. \end{aligned} \tag{$**$}$$

Hence $\varphi_{\mathbf{u}}(f \vee g) = \varphi_{\mathbf{u}}(g)$.

(c) if $j = t$, then we drop the last line from $(*)$ and the last two lines from $(**)$, arguing then as in (b).

The homomorphism $\varphi_{\mathbf{u}}$ is surjective: indeed, given $(\alpha_1, \dots, \alpha_t) \in \otimes^t \mathbb{R}$, choose $f \in \mathrm{Hom}(\mathbb{R}^n, \mathbb{R}) \subseteq \mathrm{FVL}(n)$ such that $f(u_i) = \alpha_i$. By Lemma 2.8, we obtain $\varphi_{\mathbf{u}}(f) = (\alpha_1, \dots, \alpha_t)$.

Let now $f \in \mathfrak{p}_{\mathbf{u}}$. We claim that, for every $1 \le j \le t-1$, we have $D_{u_j} \cdots D_{u_1} f \in \mathfrak{p}_{(u_{j+1}, \dots, u_t)}$; from this the inclusion $\mathfrak{p}_{\mathbf{u}} \subseteq \ker \varphi_{\mathbf{u}}$ is immediate. Working by induction on t, it suffices to show that $f \in \mathfrak{p}_{\mathbf{u}}$ implies $D_{u_1} f \in \mathfrak{p}_{(u_2, \dots, u_t)}$. Let $\boldsymbol{\varepsilon} = (\varepsilon_1, \dots, \varepsilon_t)$ be such that $f = 0$ on $S(\mathbf{u}, \boldsymbol{\varepsilon})$; without loss of generality $\varepsilon_1 = 1$. We prove that $D_{u_1} f = 0$ on $S((u_2, \dots, u_t), (\varepsilon_2, \dots, \varepsilon_t))$. Indeed, let

$$v = \lambda_2 \varepsilon_2 u_2 + \lambda_3(\varepsilon_2 u_2 + \varepsilon_3 u_3) + \cdots + \lambda_t(\varepsilon_2 u_2 + \cdots + \varepsilon_t u_t),$$

for certain $\lambda_2, \dots, \lambda_t \in \mathbb{R}^+$. Choose $h_0 > 0$ such that, for $0 < h \le h_0$, we have $\lambda_1(h) = 1 - h(\lambda_2 + \cdots + \lambda_t) > 0$. Then, for $0 < h \le h_0$, we obtain

$$\begin{aligned} u_1 + hv &= u_1 + h\lambda_2 \varepsilon_2 u_2 + h\lambda_3(\varepsilon_2 u_2 + \varepsilon_3 u_3) + \cdots + h\lambda_t(\varepsilon_2 u_2 + \cdots + \varepsilon_t u_t) \\ &= \lambda_1(h) u_1 + h\lambda_2(u_1 + \varepsilon_2 u_2) + h\lambda_3(u_1 + \varepsilon_2 u_2 + \varepsilon_3 u_3) + \cdots \\ &\qquad + h\lambda_t(u_1 + \varepsilon_2 u_2 + \cdots + \varepsilon_t u_t) \\ &\in S(\mathbf{u}, \boldsymbol{\varepsilon}). \end{aligned}$$

Hence,

$$D_{u_1} f(v) = \lim_{h_0 \ge h \to 0^+} \frac{f(u_1 + hv) - f(u_1)}{h} = 0,$$

and we are through, since v is an arbitrary element of $S((u_2, \dots, u_t), (\varepsilon_2, \dots, \varepsilon_t))$.

We now prove the inclusion $\ker \varphi_{\mathbf{u}} \subseteq \mathfrak{p}_{\mathbf{u}}$. Let $f \in \mathrm{FVL}(n)$ be such that

$$f(u_1) = D_{u_1} f(u_2) = \cdots = D_{u_{t-1}} \cdots D_{u_1} f(u_t) = 0,$$

and let $g = D_{u_1} f$. Then

$$g(u_2) = D_{u_2} g(u_3) = \cdots = D_{u_{t-1}} \cdots D_{u_2} g(u_t) = 0,$$

and by induction on t we may assume $g \in \mathfrak{p}_{(u_2, \dots, u_t)}$, i.e., $g = 0$ on $S((u_2, \dots, u_t), (\varepsilon_2, \dots, \varepsilon_t))$, for certain $\varepsilon_2, \dots, \varepsilon_t > 0$. By Lemma 2.3(i), g is linear on $\mathbb{R}u_1 + \mathbb{R}^+ v$, for every $v \in \mathbb{R}^n$. Since $g(u_1) = f(u_1) = 0$, it follows that g is identically 0 on the cone spanned by u_1 and

$S((u_2,\dots,u_t),(\varepsilon_2,\dots,\varepsilon_t))$, i.e., on

$$\mathbb{R}^+u_1+\mathbb{R}^+\varepsilon_2u_2+\mathbb{R}^+(\varepsilon_2u_2+\varepsilon_3u_3)+\cdots+\mathbb{R}^+(\varepsilon_2u_2+\cdots+\varepsilon_tu_t).$$

Hence, by Lemma 2.3(iv), for $\zeta>0$ sufficiently small, f is identically 0 on

$$\mathbb{R}^+u_1+\mathbb{R}^+(u_1+\zeta\varepsilon_2u_2)+\mathbb{R}^+\big(u_1+\zeta(\varepsilon_2u_2+\varepsilon_3u_3)\big)$$
$$+\cdots+\mathbb{R}^+\big(u_1+\zeta(\varepsilon_2u_2+\cdots+\varepsilon_tu_t)\big).$$

This proves $f\in\mathfrak{p}_{\mathbf{u}}$. ∎

We spell out the following corollary, which will be needed later.

COROLLARY 3.2. *Let* $f\in\mathrm{FVL}(n)$, $\mathbf{u}=(u_1,\dots,u_t)$, $g=D_{u_{t-1}}\cdots D_{u_1}f$, $h=D_{u_t}g$. *The following are equivalent*:

1. $f\in\mathfrak{p}_{\mathbf{u}}$;
2. $f(u_1)=D_{u_1}f(u_2)=D_{u_2}D_{u_1}f(u_3)=\cdots=D_{u_{t-1}}\cdots D_{u_1}f(u_t)=0$;
3. $g(u_i)=0$ *for every* $1\le i\le t$;
4. $h(u_i)=0$ *for every* $1\le i\le t$;
5. $g=0$ *on* $\mathbb{R}u_1+\cdots+\mathbb{R}u_{t-1}+\mathbb{R}^+u_t$;
6. $h=0$ *on* $\mathbb{R}u_1+\cdots+\mathbb{R}u_{t-1}+\mathbb{R}u_t$;
7. *for every* $v\in\mathbb{R}^n$, h *has constant value* $h(v)$ *on* $\mathbb{R}u_1+\cdots+\mathbb{R}u_t+v$.

Proof. (1) is equivalent to (2) by Theorem 3.1. (2)–(6) are all equivalent by Corollary 2.7(i) and (ii). (7) implies (6) by taking $v=0$. (6) implies (7) by Corollary 2.7(ii). ∎

Our aim is now to show that every epimorphism from $\mathrm{FVL}(n)$ onto a totally ordered vector lattice coincides—up to isomorphism—with some $\varphi_{\mathbf{u}}$; in other words, that the prime ideals of $\mathrm{FVL}(n)$ are exactly the various $\mathfrak{p}_{\mathbf{u}}$'s. In the course of the proof, we will determine the structure of the germinal ideal (see Definition 3.7) associated to each prime ideal of $\mathrm{FVL}(n)$. We first generalize the construction of $C(u,\varepsilon)$ as follows: let $\boldsymbol{\varepsilon}=(\varepsilon_1,\dots,\varepsilon_t)$, with $\varepsilon_i>0$ for every i. We let $C(\mathbf{u},\boldsymbol{\varepsilon})$ be the cone spanned by

$$S\big(\mathbf{u},(1,\varepsilon_1,\dots,\varepsilon_{t-1})\big)\cup C(u_1+\varepsilon_1u_2+\cdots+\varepsilon_{t-1}u_t,\varepsilon_t).$$

Note that if $\mathbf{u}=(u)$ and $\boldsymbol{\varepsilon}=(\varepsilon)$, then $C(\mathbf{u},\boldsymbol{\varepsilon})=C(u,\varepsilon)$.

Let now

$$\mathfrak{g}_{\mathbf{u}}=\{f\in\mathrm{FVL}(n):\text{there exists }\boldsymbol{\varepsilon}\text{ such that }f=0\text{ on }C(\mathbf{u},\boldsymbol{\varepsilon})\}.$$

For a fixed $\mathbf{u}$, the set of all $C(\mathbf{u}, \boldsymbol{\varepsilon})$, as $\boldsymbol{\varepsilon}$ varies, is lower directed with respect to inclusion (easy proof). Hence $\mathfrak{g}_{\mathbf{u}}$ is an ideal of $\mathrm{FVL}(n)$, and $\mathfrak{g}_{\mathbf{u}} \cap \mathrm{F}\ell(n)$ (which we still denote by $\mathfrak{g}_{\mathbf{u}}$) is an ideal of $\mathrm{F}\ell(n)$.

LEMMA 3.3. *For every $f \in \mathrm{FVL}(n)$ and every $\mathbf{u}$ there exists $\boldsymbol{\varepsilon}$ such that*

$$f = D_{u_1} f = D_{u_2} D_{u_1} f = \cdots = D_{u_t} \cdots D_{u_1} f$$

on $C(\mathbf{u}, \boldsymbol{\varepsilon})$.

Proof. From Corollary 2.7(iv), there exist $\delta_1, \dots, \delta_t > 0$ such that $f = D_{u_1} f$ on $C(u_1, \delta_1)$ and, for every $2 \le i \le t$, $D_{u_{i-1}} \cdots D_{u_1} f = D_{u_i} D_{u_{i-1}} \cdots D_{u_1} f$ on $\mathbb{R}u_1 + \cdots + \mathbb{R}u_{i-1} + C(u_i, \delta_i)$. A trivial computation shows now that we can find $\boldsymbol{\varepsilon}$ small enough to have

$$C(\mathbf{u}, \boldsymbol{\varepsilon}) \subseteq C(u_1, \delta_1) \cap \bigcap_{2 \le i \le t} \left(\mathbb{R}u_1 + \cdots + \mathbb{R}u_{i-1} + C(u_i, \delta_i)\right).$$

∎

COROLLARY 3.4. *Let $\mathfrak{a} \supseteq \mathfrak{g}_{\mathbf{u}}$ be an ideal of $\mathrm{FVL}(n)$ and let $f \in \mathrm{FVL}(n)$. Then the set $F = \{f, D_{u_1} f, D_{u_2} D_{u_1} f, \dots, D_{u_t} \dots D_{u_1} f\}$ is either disjoint with $\mathfrak{a}$ or contained in $\mathfrak{a}$. The same statement holds if $\mathrm{FVL}(n)$ is replaced by $\mathrm{F}\ell(n)$.*

Proof. From Lemma 3.3, if $g, h \in F$, then $g - h \in \mathfrak{g}_{\mathbf{u}} \subseteq \mathfrak{a}$. Therefore, if some element of F is in $\mathfrak{a}$, then all the others are in $\mathfrak{a}$ as well. ∎

THEOREM 3.5. *Fix $\mathbf{u}$ and let $0 \le f \in \mathrm{FVL}(n)$. The following are equivalent:*

1. $f \in \mathfrak{g}_{\mathbf{u}}$;
2. *there exists $0 < g \notin \mathfrak{p}_{\mathbf{u}}$ such that $f \wedge g = 0$;*
3. $f(u_1) = D_{u_1} f(u_2) = D_{u_2} D_{u_1} f(u_3) = \cdots = D_{u_{t-2}} \cdots D_{u_1} f(u_{t-1}) = 0$, *and there exists $\varepsilon > 0$ such that $D_{u_{t-1}} \cdots D_{u_1} f \restriction C(u_t, \varepsilon) = 0$;*
4. $D_{u_t} \cdots D_{u_1} f = 0$.

If $f \in \mathrm{F}\ell(n)$ and the above conditions hold, then in (2) we can take $g \in \mathrm{F}\ell(n)$.

Proof. (1) ⇒ (2) Let $\boldsymbol{\varepsilon}$ be such that $f = 0$ on $C(\mathbf{u}, \boldsymbol{\varepsilon})$, and let D be the topological interior of $C(\mathbf{u}, \boldsymbol{\varepsilon})$. Then D is convex and n-dimensional. Choose a complete simplicial fan Σ such that f is linear on each cone of Σ; if $f \in \mathrm{F}\ell(n)$, we take Σ to be unimodular as well. Let $\Delta = \{\sigma \in \Sigma(n) : \sigma \cap D \neq \emptyset\}$, $|\Delta| = \bigcup\{\sigma : \sigma \in \Delta\} \supseteq D$. If $\sigma \in \Delta$, then $\sigma \cap D$ is n-dimensional; hence $f = 0$ on $|\Delta|$. Let $A \supseteq D$ be the topological interior of $|\Delta|$. We can easily construct $g \in \mathrm{FVL}(n)$ such that $g(v) > 0$ for every

$v \in A$, while $g(v) = 0$ for every $v \in \mathbb{R}^n \setminus A$; moreover, if Σ is unimodular, then we can take $g \in \mathrm{F}\ell(n)$. It is clear that $f \wedge g = 0$, and we need only prove that $g \notin \mathfrak{p}_{\mathbf{u}}$. Suppose not. Then we can choose $\boldsymbol{\delta} = (\delta_1, \ldots, \delta_t)$ small enough to have $g = 0$ on $S(\mathbf{u}, \boldsymbol{\delta}) \subseteq C(\mathbf{u}, \boldsymbol{\varepsilon})$. By definition of $C(\mathbf{u}, \boldsymbol{\varepsilon})$, there exists $v \in S(\mathbf{u}, \boldsymbol{\delta}) \cap D \subseteq A$. But then $g(v) > 0$, which is impossible.

(2) $\Rightarrow$ (3) Let $0 < g \notin \mathfrak{p}_{\mathbf{u}}$ with $f \wedge g = 0$; we work by induction on t. If $t = 1$ then, since g is continuous, there must exist ε_1 with $g > 0$ on $C(u_1, \varepsilon_1) \setminus \{0\}$. Hence $f \restriction C(u_1, \varepsilon_1) = 0$, which is what we wanted to prove. Let $t > 1$, and let $\mathbf{v} = (u_1, \ldots, u_{t-1})$; then $\mathfrak{p}_{\mathbf{v}} \supset \mathfrak{p}_{\mathbf{u}}$. If $g \notin \mathfrak{p}_{\mathbf{v}}$, then by inductive hypothesis $f(u_1) = D_{u_1} f(u_2) = \cdots = D_{u_{t-3}} \cdots D_{u_1} f(u_{t-2}) = 0$ and $D_{u_{t-2}} \cdots D_{u_1} f \restriction C(u_{t-1}, \delta) = 0$ for some $\delta > 0$. By Lemma 2.3(v), $D_{u_{t-1}} \cdots D_{u_1} f$ is identically 0 over $\mathbb{R}^n$, and we are through. If, on the other hand, $g \in \mathfrak{p}_{\mathbf{v}} \setminus \mathfrak{p}_{\mathbf{u}}$, then by Corollary 3.2 $g(u_1) = D_{u_1} g(u_2) = \cdots = D_{u_{t-2}} \cdots D_{u_1} g(u_{t-1}) = 0$, while $D_{u_{t-1}} \cdots D_{u_1} g(u_t) > 0$. We claim that $f(u_1) = D_{u_1} f(u_2) = \cdots = D_{u_{t-2}} \cdots D_{u_1} f(u_{t-1}) = 0$. Suppose not, and take the first i such that $D_{u_{i-1}} \cdots D_{u_1} f(u_i) > 0$ (here $1 \le i \le t-1$, and $i = 1$ means $f(u_1) > 0$). Then

$$\begin{aligned} 0 &= D_{u_i} D_{u_{i-1}} \cdots D_{u_1}(f \wedge g) \qquad (\text{since } f \wedge g = 0) \\ &= D_{u_i}\big(D_{u_{i-1}} \cdots D_{u_1} f \wedge D_{u_{i-1}} \cdots D_{u_1} g\big) \qquad \big(\text{by Lemma 2.9(ii)}\big) \\ &= D_{u_i} D_{u_{i-1}} \cdots D_{u_1} g \qquad \big(\text{by Lemma 2.9(iii)}\big), \end{aligned}$$

which contradicts $D_{u_{t-1}} \cdots D_{u_1} g(u_t) > 0$. From our claim, we infer

$$\begin{aligned} 0 &= D_{u_{t-1}} \cdots D_{u_1}(f \wedge g) \\ &= \big(D_{u_{t-1}} \cdots D_{u_1} f\big) \wedge \big(D_{u_{t-1}} \cdots D_{u_1} g\big) \end{aligned}$$

and, since $D_{u_{t-1}} \cdots D_{u_1} g(u_t) > 0$, we conclude that $D_{u_{t-1}} \cdots D_{u_1} f$ equals 0 on some $C(u_t, \varepsilon)$.

(3) $\Rightarrow$ (1) Induction on t; if $t = 1$, then $f \restriction C(u_1, \varepsilon) = 0$, and $f \in \mathfrak{g}_{u_1}$ by definition. Assume $t > 1$ and let $h = D_{u_1} f$. Then by inductive hypothesis there exists $\boldsymbol{\delta} = (\delta_2, \ldots, \delta_t)$ such that, writing $\mathbf{v}$ for $(u_2, \ldots, u_t)$, we have $h \restriction C(\mathbf{v}, \boldsymbol{\delta}) = 0$. Now, $h(u_1) = f(u_1) = 0$, and from Lemma 2.7(ii) we see that $h = 0$ on $\mathbb{R}u_1 + C(\mathbf{v}, \boldsymbol{\delta})$. From Lemma 2.3(iv), there exists μ such that $f = 0$ on

$$C(u_1, \mu) \cap \big[\mathbb{R}u_1 + C(\mathbf{v}, \boldsymbol{\delta})\big].$$

It is a trivial matter to verify that this latter set contains a set of the form $C(\mathbf{u}, \boldsymbol{\varepsilon})$.

(3) $\Leftrightarrow$ (4) By Lemma 2.3(v) and Corollary 2.7(i). ∎

COROLLARY 3.6. *Both in* $\mathrm{FVL}(n)$ *and in* $\mathrm{F}\ell(n)$, *we have* $D_{u_t} \cdots D_{u_1} f = D_{u_t} \cdots D_{u_1} g$ *iff* $f/\mathfrak{g}_{\mathbf{u}} = g/\mathfrak{g}_{\mathbf{u}}$.

DEFINITION 3.7. Let G be an ℓ-group; $\operatorname{Spec} G$ is the set of prime ideals of G, ordered by inclusion and endowed with the Zariski topology. It is well known that $\operatorname{Spec} G$, as a poset, is a *root system* (i.e., the elements greater than any given element form a chain). If $\mathfrak{p} \in \operatorname{Spec} G$, then the *germinal* of $\mathfrak{p}$ is the ideal $\nu_{\mathfrak{p}}$ of G which is defined by any of the following equivalent conditions [BKW77, Proposition 10.5.3]:

1. $\nu_{\mathfrak{p}} = \{a \in G : \text{there exists } b \in G^+ \setminus \mathfrak{p} \text{ with } |a| \wedge b = 0\}$;
2. $\nu_{\mathfrak{p}} = \{a \in G : \{\mathfrak{q} \in \operatorname{Spec} G : a \in \mathfrak{q}\} \text{ is a neighborhood of } \mathfrak{p}\}$;
3. $\nu_{\mathfrak{p}} = \bigcap\{\mathfrak{q} \in \operatorname{Spec} G : \mathfrak{q} \subseteq \mathfrak{p}\}$;
4. $\nu_{\mathfrak{p}}$ is the only ideal of G that satisfies the following: for every $\mathfrak{q} \in \operatorname{Spec} G$, $\mathfrak{q}$ is comparable (w.r.t. inclusion) with $\mathfrak{p}$ iff $\mathfrak{q} \supseteq \nu_{\mathfrak{p}}$;
5. $\nu_{\mathfrak{p}}$ is the smallest ideal of G such that $G/\nu_{\mathfrak{p}} \simeq G/\mathfrak{p} \otimes \mathfrak{p}/\nu_{\mathfrak{p}}$.

The quotient $G/\nu_{\mathfrak{p}}$ is the ℓ-group of germs of functions at $\mathfrak{p}$, and may be interpreted as the localization of G at $\mathfrak{p}$ [BKW77, Definition 10.5.6]. From Theorem 3.5 we see that the germinal of $\mathfrak{p}_{\mathbf{u}}$ is $\mathfrak{g}_{\mathbf{u}}$, both in $\mathrm{FVL}(n)$ and in $\mathrm{F}\ell(n)$.

The following Theorem 3.8 provides us with an explicit description of $\operatorname{Spec} \mathrm{FVL}(n)$. In the next section we will develop a corresponding picture for $\mathrm{F}\ell(n)$. We define the *height* of the prime ideal $\mathfrak{p}$ in the ℓ-group G to be the supremum of all h for which there exists a strictly increasing sequence of primes

$$\mathfrak{p}_0 \subset \mathfrak{p}_1 \subset \cdots \subset \mathfrak{p}_h = \mathfrak{p}.$$

The *dimension* of G is $\dim G = \sup\{\text{height } \mathfrak{p} : \mathfrak{p} \in \operatorname{Spec} G\}$.

For $U \subseteq \mathbb{R}^n$, $U^\perp$ is the subspace of all $v \in \mathbb{R}^n$ that are orthogonal to every $u \in U$. If $\mathbf{u} = (u_1, \dots, u_t)$, we abbreviate $\{u_1, \dots, u_t\}^\perp$ with $\mathbf{u}^\perp$; therefore $\mathbf{u}^{\perp\perp} = \mathbb{R}u_1 + \cdots + \mathbb{R}u_t$. If $\mathbf{u} = (u_1, \dots, u_t)$ and $v \in \mathbb{R}^n$, then $\mathbf{u} * v$ denotes $(u_1, \dots, u_t, v)$. For $f \in \mathrm{FVL}(n)$, Zf is the *zero-set* of f, i.e., $Zf = \{v \in \mathbb{R}^n : f(v) = 0\}$.

THEOREM 3.8. *The mapping* $\mathbf{u} \mapsto \mathfrak{p}_{\mathbf{u}}$ *is an order-reversing bijection from the poset of orthonormal tuples* (*ordered by* $\mathbf{u} \le \mathbf{v}$ *iff* $\mathbf{u}$ *is an initial segment of* $\mathbf{v}$) *onto* $\operatorname{Spec} \mathrm{FVL}(n)$. *The mapping* $\mathfrak{p}_{\mathbf{u}} \mapsto \mathfrak{g}_{\mathbf{u}}$ *associates to every prime its germinal, and is an order-reversing bijection between the root system of prime ideals and the poset of germinal ideals. If* $\mathbf{u} = (u_1, \dots, u_t)$, *then* height $\mathfrak{p}_{\mathbf{u}} = n - t$, $\dim \mathrm{FVL}(n)/\mathfrak{p}_{\mathbf{u}} = t - 1$, *and* height $\mathfrak{p}_{\mathbf{u}} + \dim \mathrm{FVL}(n)/\mathfrak{p}_{\mathbf{u}} = n - 1$

$= \dim \mathrm{FVL}(n)$. *We have*

$$\frac{\mathfrak{p}_{\mathbf{u}}}{\mathfrak{g}_{\mathbf{u}}} \simeq \text{ the subvector lattice } \left\{D_{u_t} \cdots D_{u_1} f : f \in \mathfrak{p}_{\mathbf{u}}\right\} \text{ of } \mathrm{FVL}(n)$$
$$\simeq \mathrm{FVL}(n-t),$$

and hence the localization of $\mathrm{FVL}(n)$ *at* $\mathfrak{p}_{\mathbf{u}}$ *is*

$$\frac{\mathrm{FVL}(n)}{\mathfrak{g}_{\mathbf{u}}} \simeq \frac{\mathrm{FVL}(n)}{\mathfrak{p}_{\mathbf{u}}} \otimes \frac{\mathfrak{p}_{\mathbf{u}}}{\mathfrak{g}_{\mathbf{u}}}$$
$$\simeq \otimes^t \mathbb{R} \otimes \mathrm{FVL}(n-t).$$

Proof. The mapping $\mathbf{u} \mapsto \mathfrak{p}_{\mathbf{u}}$ is clearly injective and order-reversing. Let $\mathfrak{p}$ be any prime ideal and let $W = S^{n-1} \cap \bigcap\{Zf : f \in \mathfrak{p}\}$. It cannot be true that $W = \emptyset$, for otherwise by the compactness of S^{n-1} the proper ideal $\mathfrak{p}$ would contain a strong unit (i.e., a cofinal element of $\mathrm{FVL}(n)^+$). Say $u \in W$, so that $\mathfrak{p} \subseteq \mathfrak{p}_{(u)}$. Fix the greatest t such that $\mathfrak{p} \subseteq \mathfrak{p}_{\mathbf{u}}$, for some $\mathbf{u} = (u_1, \ldots, u_t)$. We already noted that the germinal of $\mathfrak{p}_{\mathbf{u}}$ is $\mathfrak{g}_{\mathbf{u}}$. By Definition 3.7(3) $\mathfrak{g}_{\mathbf{u}} \subseteq \mathfrak{p}$, and by Corollary 3.4 $\{D_{u_t} \cdots D_{u_1} f : f \in \mathfrak{p}\} \subseteq \mathfrak{p}$. Let

$$V = S^{n-1} \cap \mathbf{u}^{\perp} \cap \bigcap\left\{ZD_{u_t} \cdots D_{u_1} f : f \in \mathfrak{p}\right\}.$$

We must have $V = \emptyset$, for otherwise, if $v \in V$, then we would have $\mathfrak{p} \subseteq \mathfrak{p}_{\mathbf{u} * v}$ by Corollary 3.2. By the compactness of $S^{n-1} \cap \mathbf{u}^{\perp}$, there exist $f_1, \ldots, f_q \in \mathfrak{p}$ with

$$S^{n-1} \cap \mathbf{u}^{\perp} \cap ZD_{u_t} \cdots D_{u_1} f_1 \cap \cdots \cap ZD_{u_t} \cdots D_{u_1} f_q = \emptyset.$$

Let

$$g = \bigvee_{1 \le j \le q} |D_{u_t} \cdots D_{u_1} f_j| \in \mathfrak{p}.$$

Then $g = 0$ on $\mathbf{u}^{\perp\perp}$ (since every $D_{u_t} \cdots D_{u_1} f_j$ is 0 on $\mathbf{u}^{\perp\perp}$) and $g > 0$ on every point of $\mathbf{u}^{\perp} \setminus \{0\}$. By Corollary 3.2(7), every $D_{u_t} \cdots D_{u_1} f_j$ is constant on $\mathbf{u}^{\perp\perp} + v$, for every $v \in \mathbb{R}^n$. By using the direct sum decomposition $\mathbb{R}^n = \mathbf{u}^{\perp\perp} \oplus \mathbf{u}^{\perp}$, we see that $g > 0$ on every point of $\mathbb{R}^n \setminus \mathbf{u}^{\perp\perp}$. Let f be any element of $\mathfrak{p}_{\mathbf{u}}$. By Corollary 3.2, $D_{u_t} \cdots D_{u_1} f = 0$ on $\mathbf{u}^{\perp\perp}$, and therefore $Zg \subseteq Z|D_{u_t} \cdots D_{u_1} f|$. It is well known that, in this case, there exists a positive integer m such that $mg \geq |D_{u_t} \cdots D_{u_1} f|$ [Bak68, Lemma 3.3; Mad85, Lemma 1.4]. We conclude that $D_{u_t} \cdots D_{u_1} f \in \mathfrak{p}$ and, again by Corollary 3.4, $f \in \mathfrak{p}$. Since f was arbitrary, we obtain $\mathfrak{p} = \mathfrak{p}_{\mathbf{u}}$.

The mapping $\mathfrak{p}_{\mathbf{u}} \mapsto \mathfrak{g}_{\mathbf{u}}$ is injective and order-reversing by Theorem 3.5. The statements about height and dimension follow from the order anti-isomorphism $\mathbf{u} \mapsto \mathfrak{p}_{\mathbf{u}}$.

Let us now extend $u_1, \dots, u_t$ to an orthonormal basis $u_1, \dots, u_t, u_{t+1}, \dots, u_n$ of $\mathbb{R}^n$. We perform a linear change of coordinates, so that each element of FVL(n) is expressed as in Definition 1.1(3), with the projection x_i giving the u_ith coordinate. Then the restriction mapping $f \mapsto f \upharpoonright \mathbf{u}^{\perp}$ corresponds to setting every x_i, for $i \leq t$, to 0, and is a vector lattice epimorphism ψ: FVL$(n) \to$ FVL$(n-t)$. Let χ: $\mathfrak{p}_{\mathbf{u}} \to$ FVL(n) be defined by $\chi(f) = D_{u_t} \cdots D_{u_1} f$ and let H be the image of χ. By Lemma 2.9(ii) and Corollary 3.2 χ is a vector lattice homomorphism, and by Theorem 3.5 $\ker \chi = \mathfrak{g}_{\mathbf{u}}$. We conclude that $H \simeq \mathfrak{p}_{\mathbf{u}}/\mathfrak{g}_{\mathbf{u}}$. By Corollary 3.2(7), every $h \in H$ is constant on $\mathbf{u}^{\perp\perp} + v$, for every $v \in \mathbb{R}^n$; this shows that $\psi \upharpoonright H$: $H \to$ FVL$(n-t)$ is bijective, and hence an isomorphism. Our last statement follows from Definition 3.7(5). ∎

4. PRIME IDEALS IN FREE ℓ-GROUPS

In passing from FVL(n) to F$\ell(n)$, the main point at issue is that, while for every $u \in S^{n-1}$ there exists $f \in$ FVL(n) and a neighborhood (w.r.t. S^{n-1}) U of u with $f(u) = 0$ and $f > 0$ in $U \setminus \{u\}$, such an f may not exist in F$\ell(n)$. The simplest example is for $n = 2$; any $f \in$ F$\ell(2)$ which is 0 in $u = (\sqrt{3}/2, 1/2) \in S^1$ must be 0 in some neighborhood of u.

The function $\varphi_{\mathbf{u}}$: FVL$(n) \to \otimes^t \mathbb{R}$ defined at the beginning of Section 3 is still a homomorphism if restricted to F$\ell(n)$ (of course, it is not surjective). For simplicity's sake, we continue to denote its kernel $\mathfrak{p}_{\mathbf{u}} \cap$ F$\ell(n)$ by $\mathfrak{p}_{\mathbf{u}}$. Theorem 3.5 ensures that, even in F$\ell(n)$, the germinal of $\mathfrak{p}_{\mathbf{u}}$ is $\mathfrak{g}_{\mathbf{u}}$. Hence, the same argument used in the proof of Theorem 3.8 shows that every prime ideal of F$\ell(n)$ is of the form $\mathfrak{p}_{\mathbf{u}}$, and $\mathfrak{p}_{\mathbf{u}} \mapsto \mathfrak{g}_{\mathbf{u}}$ is still an anti-isomorphism between the root system of prime ideals and the poset of germinals.

On the other hand, the order-reversing surjection $\mathbf{u} \mapsto \mathfrak{p}_{\mathbf{u}}$ from the poset of orthonormal tuples to Spec F$\ell(n)$ is not injective any more. For example, if $n = 2$, $u = (\sqrt{3}/2, 1/2)$, and $v = (-1/2, \sqrt{3}/2)$, then $\mathfrak{p}_{(u,v)} = \mathfrak{p}_{(u)} = \mathfrak{g}_{(u)} = \mathfrak{g}_{(u,v)} =$ {functions in F$\ell(2)$ that are 0 in a neighborhood of u}. The following definition aims at re-establishing injectiveness. Let us first recall that a subspace of $\mathbb{R}^n$ has a basis of elements in $\mathbb{Z}^n$ iff it is definable via linear equations with integer coefficients.

DEFINITION 4.1. Let $\mathbf{u} = (u_1, \dots, u_t)$ be an orthonormal tuple. We let $U_{\mathbf{u}}$ be the subspace of $\mathbb{R}^n$ defined by

$$U_{\mathbf{u}} = (\mathbf{u}^{\perp} \cap \mathbb{Z}^n)^{\perp}.$$

The *rank* of $\mathbf{u}$, denoted $\operatorname{rank}(\mathbf{u})$, is the dimension of $U_{\mathbf{u}}$ as a real vector space (equivalently, the rank of $U_{\mathbf{u}} \cap \mathbb{Z}^n$ as a free $\mathbb{Z}$-module). The *$\mathbb{Z}$-reduction* of $\mathbf{u}$, denoted $\operatorname{red}(\mathbf{u})$, is the orthonormal tuple defined by induction on t as follows:

- if $t = 1$, then $\operatorname{red}(\mathbf{u}) = \mathbf{u}$;
- if $t > 1$ and $\mathbf{u} = \mathbf{v} * w$, write $w = p + q$, with $p \in U_{\operatorname{red}(\mathbf{v})}$ and $q \in U^{\perp}_{\operatorname{red}(\mathbf{v})}$; if $q = 0$, define $\operatorname{red}(\mathbf{u}) = \operatorname{red}(\mathbf{v})$, while if $q \neq 0$, define $\operatorname{red}(\mathbf{u}) = \operatorname{red}(\mathbf{v}) * (q/\|q\|)$.

If $\operatorname{red}(\mathbf{u}) = \mathbf{u}$, then we say that $\mathbf{u}$ is *$\mathbb{Z}$-reduced*. If $t = 1$, say $\mathbf{u} = (u)$ with $u = (u^1, \dots, u^n)$, then we write U_u and $\operatorname{rank}(u)$ for $U_{\mathbf{u}}$ and $\operatorname{rank}(\mathbf{u})$. This definition of rank is sensible, because $\operatorname{rank}(u)$ coincides with the rank of the free abelian group generated by $u^1, \dots, u^n$ in $\mathbb{R}$ [see Theorem 4.7(ii)].

LEMMA 4.2. *We have*:

(i) $U_{\mathbf{u}} = \bigcap\{\ker f : f \upharpoonright \mathbb{Z}^n \in \operatorname{Hom}(\mathbb{Z}^n, \mathbb{Z})$ *and* $\mathbf{u}^{\perp\perp} \subseteq \ker f\}$ = *the intersection of all subspaces of* $\mathbb{R}^n$ *that contain* $u_1, \dots, u_t$ *and are definable via equations with integer coefficients*;

(ii) $\mathbf{u}^{\perp} \cap \mathbb{Z}^n = \operatorname{red}(\mathbf{u})^{\perp} \cap \mathbb{Z}^n$ (*and hence* $U_{\mathbf{u}} = U_{\operatorname{red}(\mathbf{u})}$);

(iii) *for every* $f \in \operatorname{Hom}(\mathbb{Z}^n, \mathbb{Z})$, $f \upharpoonright \mathbf{u}^{\perp\perp} = 0$ *iff* $f \upharpoonright \operatorname{red}(\mathbf{u})^{\perp\perp} = 0$;

(iv) $U_{\mathbf{u}}^{\perp} \cap \mathbb{Z}^n = \mathbf{u}^{\perp} \cap \mathbb{Z}^n$;

(v) *if* $\mathbf{u} = (u_1, \dots, u_t)$ *is $\mathbb{Z}$-reduced, then* $U_{\mathbf{u}}$ *is the direct sum*

$$U_{\mathbf{u}} = U_{u_1} \oplus U_{u_2} \oplus \cdots \oplus U_{u_t},$$

and in particular, $\operatorname{rank}(\mathbf{u}) = \sum_{1 \le i \le t} \operatorname{rank}(u_i)$.

Proof. The identification of $\mathbb{Z}^n$ with its dual makes (i) obvious. We prove (ii) by induction on t. If $t = 1$, then the statement is trivial. Let $t > 1$ and let $\mathbf{u} = \mathbf{v} * w$. By inductive hypothesis, $\mathbf{v}^{\perp} \cap \mathbb{Z}^n = \operatorname{red}(\mathbf{v})^{\perp} \cap \mathbb{Z}^n$. Decompose w as in Definition 4.1.

Case 1. $q = 0$. We need to prove $\mathbf{u}^{\perp} \cap \mathbb{Z}^n = \operatorname{red}(\mathbf{v})^{\perp} \cap \mathbb{Z}^n$. The $\subseteq$ inclusion is clear. Let $z \in \operatorname{red}(\mathbf{v})^{\perp} \cap \mathbb{Z}^n$. Then $z \in \mathbf{v}^{\perp} \cap \mathbb{Z}^n$ by inductive hypothesis, and z is orthogonal to w because $q = 0$ yields $w \in (\operatorname{red}(\mathbf{v})^{\perp} \cap \mathbb{Z}^n)^{\perp}$.

Case 2. $q \neq 0$. We need to prove $\mathbf{v}^{\perp} \cap w^{\perp} \cap \mathbb{Z}^n = \operatorname{red}(\mathbf{v})^{\perp} \cap q^{\perp} \cap \mathbb{Z}^n$. Let $z \in \mathbf{v}^{\perp} \cap w^{\perp} \cap \mathbb{Z}^n$. Then $z \in \operatorname{red}(\mathbf{v})^{\perp}$ by inductive hypothesis. Also, $0 = \langle z, w \rangle = \langle z, p \rangle + \langle z, q \rangle = \langle z, q \rangle$, and hence $z \in q^{\perp}$ ($\langle -, - \rangle$ denotes dot product). One proves similarly the $\supseteq$ inclusion.

(iii) follows clearly from (ii), and (iv) is straightforward from the definitions. We prove (v) working by induction on t; it suffices to show that

$U_{\mathbf{u}} = U_{\mathbf{v}} \oplus U_w$, under the hypothesis $\mathbf{u} = \mathbf{v} * w$. By definition, both $U_{\mathbf{v}}$ and U_w are contained in $U_{\mathbf{u}}$. Since $\mathbf{u}$ is $\mathbb{Z}$-reduced, $U_{\mathbf{v}}^{\perp}$ is a subspace of $\mathbb{R}^n$ that contains w and is definable via integer equations. By (i), $U_w \subseteq U_{\mathbf{v}}^{\perp}$, and hence $U_{\mathbf{v}} \cap U_w = \{0\}$. Look at $U_{\mathbf{v}} + U_w$; as a subspace of $\mathbb{R}^n$, it has a basis of vectors in $\mathbb{Z}^n$. Therefore, it is definable via integer equations, and we have

$$\begin{aligned} U_{\mathbf{v}} + U_w &= \bigcap \left\{ z^{\perp} : z \in \mathbb{Z}^n, U_{\mathbf{v}} \subseteq z^{\perp}, U_w \subseteq z^{\perp} \right\} \\ &= \bigcap \left\{ z^{\perp} : z \in \mathbb{Z}^n \cap \mathbf{v}^{\perp} \cap w^{\perp} \right\} \\ &= U_{\mathbf{u}}. \end{aligned}$$

∎

Note that in general $U_{\mathbf{u}} \cap \mathbb{Z}^n$ is not the direct sum of $U_{u_i} \cap \mathbb{Z}^n$, for $1 \le i \le t$; take $n = t = 2$, with $u_1 = (1/\sqrt{2}, 1/\sqrt{2})$ and $u_2 = (-1/\sqrt{2}, 1/\sqrt{2})$.

LEMMA 4.3. *Let $f \in \mathrm{F}\ell(n)$, $\mathbf{u} = (u_1, \dots, u_t)$, $h = D_{u_t} \cdots D_{u_1} f$. Then, for every $v \in \mathbb{R}^n$ and $u \in U_{\mathbf{u}}$, we have $D_{u+v} h = D_v h$. In particular, $D_u h = D_0 h = h$.*

Proof. Assume first that $f \in \mathfrak{p}_{\mathbf{u}}$; we claim that in this case, for every $v \in \mathbb{R}^n$, h has constant value $h(v)$ on $U_{\mathbf{u}} + v$. Let Σ be a complete fan such that h is linear on its cones. Let $\Sigma(n) = \{\sigma_1, \dots, \sigma_r\}$ and let $g_1, \dots, g_r \in \mathrm{Hom}(\mathbb{Z}^n, \mathbb{Z})$ be such that $h \upharpoonright \sigma_i = g_i \upharpoonright \sigma_i$. Since h is continuous and the σ_i's cover $\mathbb{R}^n$, it suffices to show that $U_{\mathbf{u}} \subseteq \ker g_i$, for every i. By Lemma 4.2(i), we just need to show $\mathbf{u}^{\perp\perp} \subseteq \ker g_i$. Pick a point p in the topological interior of σ_i. Then

$$g_i \upharpoonright \left(\sigma_i \cap \left(\mathbf{u}^{\perp\perp} + p \right) \right) = h \upharpoonright \left(\sigma_i \cap \left(\mathbf{u}^{\perp\perp} + p \right) \right).$$

By Corollary 3.2(7), h is constant on $\mathbf{u}^{\perp\perp} + p$, and so g_i is constant on $\sigma_i \cap (\mathbf{u}^{\perp\perp} + p)$. Since p is in the topological interior of $\sigma_i \in \Sigma(n)$, $\mathbf{u}^{\perp\perp}$ and $\sigma_i \cap (\mathbf{u}^{\perp\perp} + p)$ have the same affine dimension t. This implies $\ker g_i \supseteq \mathbf{u}^{\perp\perp}$, and establishes our claim. By Lemma 2.4, f satisfies $D_{u+v} f = D_v f$, for all $u \in U_{\mathbf{u}}$ and $v \in \mathbb{R}^n$.

Let now f be any element of $\mathrm{F}\ell(n)$ and let $h = D_{u_t} \cdots D_{u_1} f$. Complete $u_1, \dots, u_t$ to an orthonormal basis $u_1, \dots, u_t, u_{t+1}, \dots, u_n$ of $\mathbb{R}^n$ and let $g = D_{u_n} \cdots D_{u_{t+1}} h$; by Lemma 2.8, g belongs to $\mathrm{F}\ell(n) \cap \mathrm{Hom}(\mathbb{R}^n, \mathbb{R})$, and hence to $\mathrm{Hom}(\mathbb{Z}^n, \mathbb{Z})$. From Corollary 2.7(iii) and (i), $D_{u_t} \cdots D_{u_1}(h - g) = D_{u_t} \cdots D_{u_1} h - D_{u_t} \cdots D_{u_1} g = h - g$ and $h(u_i) = g(u_i)$ for every $1 \le i \le t$. Hence, by Corollary 3.2(4), $h - g \in \mathfrak{p}_{\mathbf{u}}$. But then $D_{u+v} h - g = D_{u+v} h - D_{u+v} g = D_{u+v}(h - g) = D_v(h - g) = D_v h - D_v g = D_v h - g$, and we cancel g from both sides of this equality. ∎

LEMMA 4.4. *If* $\mathbf{v} = (v_1, \ldots, v_q)$ *is the* $\mathbb{Z}$*-reduction of* $\mathbf{u} = (u_1, \ldots, u_t)$*, then* $\mathfrak{p}_{\mathbf{u}} = \mathfrak{p}_{\mathbf{v}}$ *in* $\mathrm{F}\ell(n)$*. If* $\mathbf{u}, \mathbf{v}$ *are both* $\mathbb{Z}$*-reduced and* $\mathfrak{p}_{\mathbf{u}} = \mathfrak{p}_{\mathbf{v}}$ *in* $\mathrm{F}\ell(n)$*, then* $\mathbf{u} = \mathbf{v}$.

Proof. Choose $f \in \mathrm{F}\ell(n)$; Lemma 4.3 and the inductive definition of $\mathbf{v} = \mathrm{red}(\mathbf{u})$ show that $D_{u_t} \cdots D_{u_1} f = D_{v_q} \cdots D_{v_1} f = h$. By Corollary 3.2(6), we need to prove that h is 0 on $\mathbf{u}^{\perp\perp}$ iff it is 0 on $\mathbf{v}^{\perp\perp}$. Complete $u_1, \ldots, u_t$ to an orthonormal basis $u_1, \ldots, u_t, u_{t+1}, \ldots, u_n$ of $\mathbb{R}^n$. Then $g = D_{u_n} \cdots D_{u_{t+1}} h \in \mathrm{Hom}(\mathbb{Z}^n, \mathbb{Z})$ is such that $g \restriction \mathbf{u}^{\perp\perp} = h \restriction \mathbf{u}^{\perp\perp}$ and $g \restriction \mathbf{v}^{\perp\perp} = h \restriction \mathbf{v}^{\perp\perp}$. By Lemma 4.2(iii), $g \restriction \mathbf{u}^{\perp\perp} = 0$ iff $g \restriction \mathbf{v}^{\perp\perp} = 0$; our first claim is thus established.

Let $\mathbf{u}$, $\mathbf{v}$ be both $\mathbb{Z}$-reduced and assume $\mathbf{u} \neq \mathbf{v}$; we prove $\mathfrak{p}_{\mathbf{u}} \neq \mathfrak{p}_{\mathbf{v}}$. Without loss of generality, $\mathbf{u} = (u_1, \ldots, u_t)$, $\mathbf{v} = (v_1, \ldots, v_q)$, and $t \leq q$. Let i be the least index such that $u_i \neq v_i$. In case $u_j = v_j$ for every $j \leq t$ (and then necessarily $t < q$), we set $u_{t+1} = 0$ and $i = t + 1$; this convention avoids working by cases. Write $\mathbf{s} = (u_1, \ldots, u_{i-1})$, and consider $U_{\mathbf{s}}$; if $i = 1$, then $U_{\mathbf{s}} = \{0\}$. Let $e_1, \ldots, e_n$ be a basis for $\mathbb{Z}^n$ such that $e_{m+1}, \ldots, e_n$ is a $\mathbb{Z}$-module basis for $U_{\mathbf{s}} \cap \mathbb{Z}^n$ (and hence a vector space basis for $U_{\mathbf{s}}$). We have $0 \leq \mathrm{rank}(\mathbf{s}) = n - m \leq n - 1$ (it cannot be true that $m = 0$, because $0 \neq v_i \in U_{\mathbf{s}}{}^{\perp}$). Write

$$u_i = \sum_{1 \leq j \leq n} a_j e_j; \qquad v_i = \sum_{1 \leq j \leq n} b_j e_j;$$

$$u_i' = \sum_{1 \leq j \leq m} a_j e_j; \qquad v_i' = \sum_{1 \leq j \leq m} b_j e_j.$$

We have $u_i' \neq v_i'$ (if not, $u_i - v_i \in U_{\mathbf{s}}$ and, since $\mathbf{u}, \mathbf{v}$ are $\mathbb{Z}$-reduced, both u_i and v_i are in $U_{\mathbf{s}}{}^{\perp}$; hence $u_i - v_i \in U_{\mathbf{s}} \cap U_{\mathbf{s}}{}^{\perp}$, and $u_i = v_i$).

As in the proof of Theorem 3.8, we perform a linear change of coordinates, writing the elements of $\mathrm{F}\ell(n)$ as (equivalence classes of) ℓ-polynomials over $x_1, \ldots, x_n$, where x_j is the projection over the e_jth coordinate. Upon identifying $\mathrm{F}\ell(m)$ with the ℓ-subgroup of $\mathrm{F}\ell(n)$ consisting of ℓ-polynomials containing $x_1, \ldots, x_m$ only, we can find $f \in \mathrm{F}\ell(m) \subseteq \mathrm{F}\ell(n)$ such that:

(i) for every $w \in \mathbb{R}^n$, f is constant over $U_{\mathbf{s}} + w$;

(ii) $f(v_i') = f(v_i) \neq 0$;

(iii) $f \restriction \{e_1, \ldots, e_m\}^{\perp\perp} = 0$ in an m-dimensional neighborhood of u_i' in $\{e_1, \ldots, e_m\}^{\perp\perp}$, and hence $f = 0$ in an n-dimensional neighborhood of u_i in $\mathbb{R}^n$ (we require this condition only in case $i \leq t$, so that $u_i \neq 0$).

Since $u_1, \ldots, u_{i-1} \in U_{\mathbf{s}}$, we see from Lemma 2.4 that $f = D_{u_1} f = D_{u_2} D_{u_1} f = \cdots = D_{u_{i-1}} \cdots D_{u_1} f$. If $i = t + 1$, then $f \in \mathfrak{p}_{\mathbf{u}}$ since $f \restriction U_{\mathbf{s}} = 0$. If $i \leq t$, then by (iii), Lemma 2.3(v), and Theorem 3.5(4), we have $f \in \mathfrak{g}_{\mathbf{u}} \subseteq \mathfrak{p}_{\mathbf{u}}$. On the other hand, Corollary 3.2(3) and $f(v_i) \neq 0$ show that $f \notin \mathfrak{p}_{\mathbf{v}}$. ∎

Let $\mathbf{u} = (u_1, \ldots, u_t)$ be a $\mathbb{Z}$-reduced tuple, and let $u_i = (u_i^1, \ldots, u_i^n)$. For every o-group G and every subset S of G, the o-subgroup generated by S in G coincides with the subgroup generated by S. By taking S to be the set of images of the generators $x_1, \ldots, x_n$ of $\mathrm{F}\ell(n)$ under

$$\varphi_{\mathbf{u}} \colon \mathrm{F}\ell(n) \to \otimes^t \mathbb{R},$$

one sees that $\varphi_{\mathbf{u}}[\mathrm{F}\ell(n)]$ is the subgroup $H_{\mathbf{u}}$ of $\otimes^t \mathbb{R}$ generated by the rows of the matrix

$$M_{\mathbf{u}} = \begin{pmatrix} u_1^1 & \cdots & u_t^1 \\ u_1^2 & \cdots & u_t^2 \\ \vdots & \cdots & \vdots \\ u_1^n & \cdots & u_t^n \end{pmatrix}.$$

Indeed, these rows are exactly the images of $x_1, \ldots, x_n$.

DEFINITION 4.5. Notation being as above, we let $\mathbb{Z}[u_i^1, \ldots, u_i^n]$ be the o-subgroup $\mathbb{Z}u_i^1 + \mathbb{Z}u_i^2 + \cdots + \mathbb{Z}u_i^n$ of $\mathbb{R}$ generated by $u_i^1, \ldots, u_i^n$. We also let G_i be the o-subgroup of $\mathbb{R}$ defined by

$$G_i = \left\{ \langle u_i, w \rangle : w \in \{u_1, \ldots, u_{i-1}\}^{\perp} \cap \mathbb{Z}^n \right\}$$

(equivalently, G_i is the o-subgroup of $\mathbb{R}$ generated by the image under $\langle u_i, - \rangle$ of any $\mathbb{Z}$-basis of $\{u_1, \ldots, u_{i-1}\}^{\perp} \cap \mathbb{Z}^n$).

Note that $G_1 = \mathbb{Z}[u_1^1, \ldots, u_1^n]$; for $i > 1$, G_i and $\mathbb{Z}[u_i^1, \ldots, u_i^n]$ may not be isomorphic as o-groups, even though they always have the same rank (see Theorem 4.7(ii) and Example 5.3). We pause a moment to give a criterion for deciding when two finitely generated o-subgroups of $\mathbb{R}$ of the same rank are isomorphic.

LEMMA 4.6. *Let $\alpha^1, \ldots, \alpha^n, \beta^1, \ldots, \beta^n \in \mathbb{R}$ be such that both $\alpha^1, \ldots, \alpha^n$ and $\beta^1, \ldots, \beta^n$ are linearly independent over $\mathbb{Q}$. Then $\mathbb{Z}[\alpha^1, \ldots, \alpha^n]$ is isomorphic to $\mathbb{Z}[\beta^1, \ldots, \beta^n]$ as o-groups iff there exists a unimodular (i.e., of determinant of absolute value 1) $n \times n$ matrix C with integer entries, and some $0 < r \in \mathbb{R}$, such that*

$$\begin{pmatrix} \beta^1 \\ \beta^2 \\ \vdots \\ \beta^n \end{pmatrix} = rC \begin{pmatrix} \alpha^1 \\ \alpha^2 \\ \vdots \\ \alpha^n \end{pmatrix}.$$

Proof. By [Fuc63, Proposition IV.2], $\mathbb{Z}[\alpha^1, \dots, \alpha^n]$ is isomorphic to $\mathbb{Z}[\beta^1, \dots, \beta^n]$ iff there exists $0 < r \in \mathbb{R}$ with $\mathbb{Z}[r\alpha^1, \dots, r\alpha^n] = \mathbb{Z}[\beta^1, \dots, \beta^n]$. This last equality holds iff there exists a unimodular matrix C with integer entries such that

$$\begin{pmatrix} \beta^1 \\ \beta^2 \\ \vdots \\ \beta^n \end{pmatrix} = C \begin{pmatrix} r\alpha^1 \\ r\alpha^2 \\ \vdots \\ r\alpha^n \end{pmatrix}.$$

∎

THEOREM 4.7. *Let* $\mathbf{u} = (u_1, \dots, u_t)$ *be* $\mathbb{Z}$*-reduced. Then*:

(i) $H_{\mathbf{u}} \simeq G_1 \otimes \cdots \otimes G_t$;

(ii) *for every* $1 \le i \le t$, *the free abelian groups* G_i *and* $\mathbb{Z}[u_i^1, \dots, u_i^n]$ *have the same rank* $r_i \ge 1$. *Also,* r_i *coincides with* $\operatorname{rank}(u_i)$, *as defined in Definition* 4.1;

(iii) $t \le \operatorname{rank}(\mathbf{u}) = r_1 + \cdots + r_t =$ (*rank of* $H_{\mathbf{u}}$ *as a free abelian group*) $\le n$.

Proof. (i) Induction on t: for $t = 1$ the statement is trivial. Let $t > 1$ and write $\mathbf{u} = \mathbf{v} * u_t$. We have an exact sequence of abelian groups

$$0 \to \ker \pi \to H_{\mathbf{u}} \xrightarrow{\pi} H_{\mathbf{v}} \to 0 \qquad (*)$$

induced by the projection

$$\pi\colon (s_1, \dots, s_t) \mapsto (s_1, \dots, s_{t-1}).$$

Since $H_{\mathbf{v}}$ is finitely generated and torsion-free, it is free, and $(*)$ splits as a sequence of abelian groups. Since $\ker \pi$ is convex in $H_{\mathbf{u}}$, $(*)$ splits also as a sequence of o-groups, and we obtain

$$H_{\mathbf{u}} \simeq H_{\mathbf{v}} \otimes \ker \pi.$$

It is clear that $\ker \pi$ and G_t are isomorphic as o-groups, and we conclude the proof by applying the inductive hypothesis.

(ii) Fix $1 < i \le t$. Choose a basis $l_1, \dots, l_n$ of $\mathbb{R}^n$ such that $l_1, \dots, l_m$ is a $\mathbb{Z}$-module basis for

$$\{u_1, \dots, u_{i-1}\}^{\perp} \cap \mathbb{Z}^n = U^{\perp}_{(u_1, \dots, u_{i-1})} \cap \mathbb{Z}^n,$$

and $l_{m+1}, \dots, l_n$ is a $\mathbb{Z}$-module basis for

$$\left(\{u_1, \dots, u_{i-1}\}^{\perp} \cap \mathbb{Z}^n\right)^{\perp} \cap \mathbb{Z}^n = U_{(u_1, \dots, u_{i-1})} \cap \mathbb{Z}^n.$$

Let T be the free abelian group of rank n generated by $l_1, \ldots, l_n$. Choose a basis $e_1, \ldots, e_n$ of $\mathbb{Z}^n$ and nonzero positive integers $m_1, \ldots, m_n$ such that $m_1 | m_2 | \cdots | m_n$ and $m_1 e_1, \ldots, m_n e_n$ is a basis for T. Then

$$m_n \mathbb{Z}^n \subseteq T \subseteq \mathbb{Z}^n,$$

and hence $G_i = \langle u_i, T \rangle$ is sandwiched between $\langle u_i, m_n \mathbb{Z}^n \rangle = m_n \mathbb{Z}[u_i^1, \ldots, u_i^n]$ and $\langle u_i, \mathbb{Z}^n \rangle = \mathbb{Z}[u_i^1, \ldots, u_i^n]$, which are both free of rank $r_i \geq 1$; hence so is G_i. An analogous stack-basis proof shows that, for every $u = (u^1, \ldots, u^n) \in \mathbb{R}^n$, the rank of u as defined in Definition 4.1 coincides with the rank of $\mathbb{Z}[u^1, \ldots, u^n]$.

(iii) It suffices to prove $\operatorname{rank}(\mathbf{u}) = r_1 + \cdots + r_t$, and this is Lemma 4.2(v). ∎

By Theorem 4.7(iii) and [Dar95, Proposition 52.15], $\operatorname{rank}(\mathbf{u})$ can be characterized as the only integer r for which there exists a free generating set $y_1, \ldots, y_n$ of $\mathrm{F}\ell(n)$ such that $y_{r+1}, \ldots, y_n \in \mathfrak{p}_{\mathbf{u}}$ and the map $f \mapsto f/\mathfrak{p}_{\mathbf{u}}$ is injective on the subgroup generated by $y_1, \ldots, y_r$. Indeed, if we choose a basis $e_1, \ldots, e_n$ of $\mathbb{Z}^n$ with $e_{r+1}, \ldots, e_n$ spanning $U_{\mathbf{u}}^{\perp}$, then the projections $y_1, \ldots, y_n$ corresponding to $e_1, \ldots, e_n$ satisfy the above requirements.

We can now prove our main theorem, that parallels Theorem 3.8 for the case of finitely generated free ℓ-groups.

THEOREM 4.8. *The mapping* $\mathbf{u} \mapsto \mathfrak{p}_{\mathbf{u}}$ *is an order-reversing bijection from the set of* $\mathbb{Z}$*-reduced tuples onto* $\operatorname{Spec} \mathrm{F}\ell(n)$. *The mapping* $\mathfrak{p}_{\mathbf{u}} \mapsto \mathfrak{g}_{\mathbf{u}}$ *associates to every prime its germinal, and is an order-reversing bijection between the root system of prime ideals and the poset of germinal ideals. If* $\mathbf{u} = (u_1, \ldots, u_t)$ *is* $\mathbb{Z}$*-reduced and has rank* r, *then* height $\mathfrak{p}_{\mathbf{u}} = n - r$, $\dim \mathrm{F}\ell(n)/\mathfrak{p}_{\mathbf{u}} = t - 1$, *and* height $\mathfrak{p}_{\mathbf{u}} + \dim \mathrm{F}\ell(n)/\mathfrak{p}_{\mathbf{u}} = n + t - r - 1 \leq n - 1 = \dim \mathrm{F}\ell(n)$. *We have*

$$\begin{aligned} \frac{\mathfrak{p}_{\mathbf{u}}}{\mathfrak{g}_{\mathbf{u}}} &\simeq \textit{the } \ell\textit{-subgroup } \left\{ D_{u_t} \cdots D_{u_1} f : f \in \mathfrak{p}_{\mathbf{u}} \right\} \textit{ of } \mathrm{F}\ell(n) \\ &\simeq \mathrm{F}\ell(n - r), \end{aligned}$$

and hence the localization of $\mathrm{F}\ell(n)$ *at* $\mathfrak{p}_{\mathbf{u}}$ *is*

$$\begin{aligned} \frac{\mathrm{F}\ell(n)}{\mathfrak{g}_{\mathbf{u}}} &\simeq \frac{\mathrm{F}\ell(n)}{\mathfrak{p}_{\mathbf{u}}} \otimes \frac{\mathfrak{p}_{\mathbf{u}}}{\mathfrak{g}_{\mathbf{u}}} \\ &\simeq G_1 \otimes \cdots \otimes G_t \otimes \mathrm{F}\ell(n - r), \end{aligned}$$

where $G_1, \ldots, G_t$ *are the o-subgroups of* $\mathbb{R}$ *introduced in Definition* 4.5.

Proof. Almost all of our statements have already been established. The standard orthonormal basis of $\mathbb{Z}^n$ yields a $\mathbb{Z}$-reduced tuple of maximal length n; hence $\dim \mathrm{F}\ell(n) = n - 1$. If $\mathbf{u}$ is $\mathbb{Z}$-reduced of rank r, then $n - r$ is both the dimension of $U_{\mathbf{u}}^{\perp}$ and the rank of $U_{\mathbf{u}}^{\perp} \cap \mathbb{Z}^n$; since $\mathbf{u}$ can be extended to a larger $\mathbb{Z}$-reduced tuple only by picking elements in $U_{\mathbf{u}}^{\perp}$, it follows that height $\mathbf{u} = n - r$. By Theorem 4.7(i) $\mathrm{F}\ell(n)/\mathfrak{p}_{\mathbf{u}}$ is isomorphic to $G_1 \otimes \cdots \otimes G_t$, and hence has dimension $t - 1$. The only claim that remains to be proved is the one about the isomorphism type of $\mathfrak{p}_{\mathbf{u}}/\mathfrak{g}_{\mathbf{u}}$. The proof is identical to that of Theorem 3.8: change coordinates to a basis $e_1, \dots, e_n$ of $\mathbb{Z}^n$ in such a way that $e_1, \dots, e_r$ is a $\mathbb{Z}$-basis for $U_{\mathbf{u}} \cap \mathbb{Z}^n$. Then

$$\psi\colon \mathrm{F}\ell(n) \to \mathrm{F}\ell(n - r)$$

given by

$$f(x_1, \dots, x_n) \mapsto f(0, \dots, 0, x_{r+1}, \dots, x_n)$$

is an epimorphism. Define

$$\chi\colon \mathfrak{p}_{\mathbf{u}} \twoheadrightarrow \chi[\mathfrak{p}_{\mathbf{u}}] = H \hookrightarrow \mathrm{F}\ell(n)$$

by $\chi(f) = D_{u_t} \cdots D_{u_1} f$. By Lemma 2.9(ii), Corollary 3.2, and Theorem 3.5, χ is an ℓ-group homomorphism with kernel $\mathfrak{g}_{\mathbf{u}}$. Therefore, $\mathfrak{p}_{\mathbf{u}}/\mathfrak{g}_{\mathbf{u}} \simeq H$. By the first part of the proof of Lemma 4.3, every $\chi(f)$ is constant on $U_{\mathbf{u}} + v$, for every $v \in \mathbb{R}^n$. Therefore $\chi(f)$ depends, as an ℓ-polynomial, on $x_{r+1}, \dots, x_n$ only, and $H \simeq \mathrm{F}\ell(n - r)$. ∎

Let us note the following nice similarity between primes in ℓ-groups and primes in integral domains. Let D be a domain that is a finitely generated algebra over a domain R. As it is well known, if R is a field, then height $\mathfrak{p} + \dim D/\mathfrak{p}$ is constant—equal to the dimension of D—as $\mathfrak{p}$ varies in the primes of D [Mat80, 14.H]. On the other hand, if R is not a field, then in general we only have trivially

$$\text{height } \mathfrak{p} + \dim D/\mathfrak{p} \leq \dim D,$$

and the sum to the left varies with the prime; for a simple example take $D = \mathbb{Z}_p[x]$, where $\mathbb{Z}_p$ is the ring of p-adic integers. Clearly, this behavior parallels that of primes in free vector lattices and free ℓ-groups, as described in Theorems 3.8 and 4.8.

As we noted in the Preliminaries, Theorem 4.8 and the categorical equivalence between ℓ-groups with strong unit and MV-algebras yield immediately a description of the prime spectrum of the finitely generated free MV-algebras. We recall that an *MV-algebra* is an algebra $A = (A,$

$\oplus, \neg, 0)$ such that $(A, \oplus, 0)$ is an abelian monoid and the identities $\neg\neg a = a$, $a \oplus (\neg 0) = \neg 0$, and $\neg(\neg a \oplus b) \oplus b = \neg(\neg b \oplus a) \oplus a$ hold. MV-algebras stand to Łukasiewicz infinite-valued logic as Boolean algebras stand to classical logic [Cha58; Mun86]; in particular, prime ideals in free MV-algebras correspond to prime theories in infinite-valued logic. Here, by a *prime theory* we mean a theory T having the property that, for every two formulas φ, ψ, either $\varphi \to \psi$ or $\psi \to \varphi$ is a theorem of T. While in classical logic prime theories coincide with complete ones, in infinite-valued logic the two notions are distinct.

Let (G, g) be an ℓ-group equipped with a fixed strong unit g. Then the structure

$$\Gamma(G, g) = ([0, g], \oplus, \neg, 0),$$

defined by

$$\begin{aligned} [0, g] &= \{a \in G : 0 \le a \le g\} \\ a \oplus b &= (a + b) \wedge g \\ \neg a &= g - a \\ 0 &= \text{the additive identity } 0 \text{ of } G, \end{aligned}$$

is an MV-algebra, and every MV-algebra is obtained from a uniquely determined ℓ-group with strong unit in such a way. The ideals of G correspond bijectively to the ideals of $\Gamma(G, g)$ via $\mathfrak{a} \mapsto \mathfrak{a} \cap [0, g]$, and $\mathfrak{a}$ is prime iff $\mathfrak{a} \cap [0, g]$ is prime. The functor $(G, g) \mapsto \Gamma(G, g)$ induces a categorical equivalence between the category of ℓ-groups with strong unit and the category of MV-algebras; see [Mun86] for the proofs of the above statements.

Let now $w_1, \ldots, w_q$ be the vertices of the unit cube $[0, 1]^n$ in $\mathbb{R}^n$, with $q = 2^n$, and let σ be the cone in $\mathbb{R}^{n+1}$ spanned by $(w_1, 1), \ldots, (w_q, 1) \in \mathbb{R}^n \times \{1\}$. Let $\mathfrak{a}$ be the principal ideal of $\mathrm{F}\ell(n + 1)$ whose elements are the functions which are 0 on σ. Then $x_{n+1}/\mathfrak{a}$ is a strong unit of $\mathrm{F}\ell(n + 1)/\mathfrak{a}$, and it is well known [Mun86] that the free MV-algebra over n generators is

$$\Gamma(\mathrm{F}\ell(n + 1)/\mathfrak{a}, x_{n+1}/\mathfrak{a}).$$

We therefore obtain the following corollary to Theorem 4.8.

COROLLARY 4.9. *The prime ideals of the free* MV*-algebra over n generators correspond bijectively to the* $\mathbb{Z}$*-reduced tuples* $\mathbf{u}$ *in* $\mathbb{R}^{n+1}$ *satisfying* $S(\mathbf{u}, \boldsymbol{\varepsilon}) \subseteq \sigma$*, for some* $\boldsymbol{\varepsilon}$*.*

5. A SIMPLE ALGORITHM FOR COMPUTING QUOTIENTS

We conclude this paper by giving a very simple recipe for computing the isomorphism types of the quotients $\mathrm{F}\ell(n)/\mathfrak{p}_{\mathbf{u}}$, given any orthonormal tuple $\mathbf{u}$ (not necessarily $\mathbb{Z}$-reduced).

The algorithm works as follows: construct the $n \times t$ matrix with real entries $M_{\mathbf{u}}$ as in Section 4. A *row operation* consists:

- either in exchanging two rows;
- or in multiplying a row by -1;
- or in adding to a row a linear combination with integer coefficients of the remaining ones.

We leave to the reader the proof of the following lemma.

LEMMA 5.1. *Given $\alpha^1, \dots, \alpha^n \in \mathbb{R}$, there exists a sequence of row operations that transform the column vector*

$$\begin{pmatrix} \alpha^1 \\ \vdots \\ \alpha^n \end{pmatrix}$$

in a column vector

$$\begin{pmatrix} \beta^1 \\ \vdots \\ \beta^r \\ 0 \\ \vdots \\ 0 \end{pmatrix},$$

where $\beta^1, \dots, \beta^r$ is any given $\mathbb{Z}$-basis for $\mathbb{Z}[\alpha^1, \dots, \alpha^n]$.

Starting from $M_1 = M_{\mathbf{u}}$, apply row operations and obtain a matrix M_1' whose first column is of the form

$$\begin{pmatrix} \alpha^1 \\ \vdots \\ \alpha^{r_1} \\ 0 \\ \vdots \\ 0 \end{pmatrix},$$

with $r_1 = \operatorname{rank}(u_1)$ and $\mathbb{Z}[u_1^1, \dots, u_1^n] = \mathbb{Z}[\alpha^1, \dots, \alpha^{r_1}]$. This corresponds to changing the standard basis of $\mathbb{Z}^n$ to a basis $e_1, \dots, e_n$ such that $e_{r_1+1}, \dots, e_n$ constitute a basis for $u_1^\perp \cap \mathbb{Z}^n$. Write $G_1 = \mathbb{Z}[\alpha^1, \dots, \alpha^{r_1}]$, and obtain M_2 by throwing away the first column and the first r_1 rows from M_1'. Repeat the process, constructing $G_2, \dots, G_t$. It is clear that the G_i's so constructed are exactly those introduced in Definition 4.5. Note that if $\mathbf{u}$ is not $\mathbb{Z}$-reduced, then some of the G_i's may be 0; in this case, we simply discard them. At any step, we can multiply column i by any nonzero positive real number. The effect is merely a cosmetic improvement of the presentation of G_i, and does not change its isomorphism type.

EXAMPLE 5.2. Let $n = 3$, $u_1 = (1/2, -1/2, 1/\sqrt{2})$, $u_2 = (0, \sqrt{2}/\sqrt{3}, 1/\sqrt{3})$, $u_3 = (-\sqrt{3}/2, -1/(2\sqrt{3}), 1/\sqrt{6})$, $\mathbf{u} = (u_1, u_2, u_3)$. Then $\mathbf{u}$ is orthonormal, $U_{u_1} = \mathbb{R}(1, -1, 0) + \mathbb{R}(0, 0, 1)$, and $U_{u_1}^\perp = \mathbb{R}(1, 1, 0)$. Since $u_2 \notin U_{u_1}^\perp$, $\mathbf{u}$ is not $\mathbb{Z}$-reduced; projecting u_2 on $U_{u_1}^\perp$ and normalizing, we obtain $v = (1/\sqrt{2}, 1/\sqrt{2}, 0)$. Since $U_v = \mathbb{R}(1, 1, 0)$, we have $U_{(u_1, v)} = U_{u_1} \oplus U_v = \mathbb{R}^3$, and therefore the $\mathbb{Z}$-reduction of $\mathbf{u}$ is (u_1, v).

We have

$$M_{\mathbf{u}} = \begin{pmatrix} \dfrac{1}{2} & 0 & -\dfrac{\sqrt{3}}{2} \\[2ex] -\dfrac{1}{2} & \dfrac{\sqrt{2}}{\sqrt{3}} & -\dfrac{1}{2\sqrt{3}} \\[2ex] \dfrac{1}{\sqrt{2}} & \dfrac{1}{\sqrt{3}} & \dfrac{1}{\sqrt{6}} \end{pmatrix}.$$

First, we make cosmetics, by multiplying the columns by 2, $\sqrt{3}$, and $2\sqrt{3}$, respectively. We obtain

$$M_1 = \begin{pmatrix} 1 & 0 & -3 \\ -1 & \sqrt{2} & -1 \\ \sqrt{2} & 1 & \sqrt{2} \end{pmatrix}.$$

We add the first row to the second, and exchange the second row with the third. The resulting matrix is

$$M_1' = \begin{pmatrix} 1 & 0 & -3 \\ \sqrt{2} & 1 & \sqrt{2} \\ 0 & \sqrt{2} & -4 \end{pmatrix}.$$

We have $G_1 = \mathbb{Z}[1, \sqrt{2}]$ and

$$M_2 = \begin{pmatrix} \sqrt{2} & -4 \end{pmatrix}.$$

We multiply the first column by $1/\sqrt{2}$, and stop the process, since M_3 is the empty matrix. We conclude that $\mathrm{F}\ell(3)/\mathfrak{p}_{\mathbf{u}}$ is isomorphic to $\mathbb{Z}[1, \sqrt{2}] \otimes \mathbb{Z}$. Since $\mathrm{rank}(\mathbf{u}) = 3$, $\mathfrak{p}_{\mathbf{u}}$ has height 0, i.e., is a minimal prime and coincides with its germinal. Note also that the dimension of $\mathrm{F}\ell(3)/\mathfrak{p}_{\mathbf{u}}$ is 1.

EXAMPLE 5.3. Let $n = 3$, $v_1 = (1, -1, 2)$, $v_2 = (1, 1 + 2\sqrt{2}, \sqrt{2})$, $\mathbf{u} = (v_1/\|v_1\|, v_2/\|v_2\|) = (u_1, u_2)$. Then $\mathrm{rank}(u_1) = 1$, $U_{u_1} = \mathbb{R}v_1$, and $\mathbf{u}$ is $\mathbb{Z}$-reduced, since $u_2 \in \mathbb{R}(1, 1, 0) + \mathbb{R}(-2, 0, 1) = U_{u_1}^{\perp}$. We have

$$M_1 = \begin{pmatrix} 1 & 1 \\ -1 & 1 + 2\sqrt{2} \\ 2 & \sqrt{2} \end{pmatrix},$$

and by row operations we obtain

$$M_1' = \begin{pmatrix} 1 & 1 \\ 0 & 2 + 2\sqrt{2} \\ 0 & -2 + \sqrt{2} \end{pmatrix}.$$

Therefore, $G_1 = \mathbb{Z}$ and

$$M_2 = \begin{pmatrix} 2 + 2\sqrt{2} \\ -2 + \sqrt{2} \end{pmatrix}.$$

We obtain $G_2 = \mathbb{Z}[2 + 2\sqrt{2}, -2 + \sqrt{2}] = \mathbb{Z}[2 - \sqrt{2}, 3\sqrt{2}]$ and $\mathrm{F}\ell(3)/\mathfrak{p}_{\mathbf{u}} \simeq \mathbb{Z} \otimes \mathbb{Z}[2 - \sqrt{2}, 3\sqrt{2}]$. Again, $\mathfrak{p}_{\mathbf{u}}$ is a minimal prime and $\dim \mathrm{F}\ell(3)/\mathfrak{p}_{\mathbf{u}} = 1$. Note that G_2 is not isomorphic to $\mathbb{Z}[u_2^1, u_2^2, u_2^3] \simeq \mathbb{Z}[1, \sqrt{2}]$. Indeed, assume that they are isomorphic. Then by Lemma 4.6 we have

$$\begin{pmatrix} 2 - \sqrt{2} \\ 3\sqrt{2} \end{pmatrix} = r\begin{pmatrix} a & b \\ c & d \end{pmatrix}\begin{pmatrix} 1 \\ \sqrt{2} \end{pmatrix},$$

for an appropriate unimodular matrix with integer entries and some real number $r > 0$. Therefore

$$2 - \sqrt{2} = r(a + b\sqrt{2}),$$
$$3\sqrt{2} = r(c + d\sqrt{2}).$$

Eliminating r and equating the rational and the irrational parts, we get

$$3a = -c + 2d,$$
$$3b = c - d.$$

Since the matrix is unimodular,

$$\begin{vmatrix} -c+2d & c-d \\ c & d \end{vmatrix} = 2d^2 - c^2 = \pm 3.$$

So c must be odd. It is impossible that

$$2d^2 - c^2 = 3,$$

because reducing mod 4 we see that d is even, and $-c^2 = 3$ has no solutions mod 8. It is also impossible that

$$2d^2 - c^2 = -3, \tag{$*$}$$

because reducing mod 4 we have

$$2d^2 = 2 \pmod 4.$$

Hence d is odd, and reducing $(*)$ mod 8 we conclude

$$2 - 1 = -3 \pmod 8,$$

which is again absurd.

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